ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (1), 190–202
doi:10.15330/cmp.16.1.190-202

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №1, С.190–202

Fractional integral operators on Hardy local Morrey spaces with variable exponents

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We establish the mapping properties of the fractional integral operators on the Hardy local Morrey spaces with variable exponents by using the extrapolation theory. The local Morrey spaces with variable exponents are generalizations and extensions of the local Morrey spaces, the Lebesgue spaces with variable exponents and the Morrey spaces with variable exponents. The Hardy local Morrey spaces with variable exponents are the Hardy spaces built on the local Morrey spaces with variable exponents. Our main result extends and generalizes the mapping properties of the fractional integral operators on the Hardy spaces, the local Morrey spaces, the Hardy spaces with variable exponents and the local Morrey spaces with variable exponents. We obtain our main result by extending the J.L. Rubio de Francia extrapolation theory to the local Morrey spaces with variable exponents. This method was originally developed by J.L. Rubio de Francia on the weighted Lebesgue spaces and recently it has been extended to a number of function spaces such as the Morrey spaces with variable exponents, the local Morrey spaces with variable exponents and the Morrey-Banach spaces. We further extend it to the local Morrey spaces with variable exponents. By using the mapping properties of the fractional integral operators on the weighted Hardy spaces, we establish the mapping properties of the fractional integral operators on the Hardy local Morrey spaces with variable exponents.

Key words and phrases: fractional integral operator, Hardy space, local Morrey space, variable exponent.

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1 Introduction

This paper aims to extend the mapping properties of the fractional integral operators to the Hardy local Morrey spaces with variable exponents.

The mapping property of the fractional integral operators on function spaces is one of the main topics in function spaces. The mapping properties of the fractional integral operator on Lebesgue spaces are well known [37, Chapter VIII, Section 4.2]. It is also named as the Hardy-Littlewood-Sobolev inequalities. The Hardy-Littlewood-Sobolev inequalities had been extended to Morrey space in [1, 28], where Morrey spaces were introduced by Morrey in [25] to study the solutions of some quasilinear elliptic partial differential equations.

The Morrey spaces had been generalized to the Morrey-Lorentz space [30, 40], the Orlicz-Morrey spaces [34] and the variable Morrey spaces (also named as the Morrey spaces with variable exponents) [2, 23]. Some extensions of Hardy-Littlewood-Sobolev inequalities on these Morrey type spaces were given in [2, 17, 29, 34, 40]. The local Morrey spaces and their

УДК 517.57
2020 *Mathematics Subject Classification:* 42B20, 42B15, 42B30, 42B35.

generalizations, the local Morrey spaces with variable exponents, are introduced and studied in [4–9, 15, 39].

Another important extension is the mapping properties of the fractional integral operators on Hardy spaces. The Hardy spaces were introduced in [36] and the mapping properties of the fractional integral operators on Hardy spaces were also given there. Further extensions were obtained in [22].

Recently, a combination of the local Morrey spaces with variable exponents and the Hardy spaces, named as the Hardy local Morrey spaces with variable exponents, was given in [18]. The boundedness of Calderón-Zygmund operators and some sublinear operators on the Hardy local Morrey spaces with variable exponents were also obtained in [18]. In view of the importance of the mapping properties of the fractional integral operators, we are motivated to establish the mapping properties of the fractional integral operators on the Hardy local Morrey spaces with variable exponents. Moreover, our main results also generalize and extend the mapping properties of the fractional integral operators obtained in [5, 7, 8, 15, 22, 36].

We use the extrapolation theory to obtain our main result. The extrapolation theory was introduced by J.L. Rubio de Francia in [31–33]. The refined extrapolation theory for Morrey type spaces were obtained in [19]. This refined version had been used in [18, 20] to study the Calderón-Zygmund operators and some sublinear operators on Hardy type spaces. In this paper, we further extend the extrapolation theory to study the fractional integral operators on the local Morrey spaces with variable exponents.

This paper is organized as follows. Section 2 contains the definition of the local Morrey spaces with variable exponents, the definition of the Hardy local Morrey spaces with variable exponents and some backgrounds for the studies of these function spaces. The main result of this paper is given in Section 3.

2 Definitions and Preliminaries

Let $\mathcal{M}$ and L^1_{loc} denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}^n$, respectively. For any $x \in \mathbb{R}^n$ and $r > 0$, let us define $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

Let $0 < \alpha < n$. The fractional integral operator is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

If $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$, then $I_\alpha : L_p \to L_q$ is bounded. The reader is referred to [26, 27] for the weighted norm inequalities of the fractional integral operators.

We use the grand maximal function to define the Hardy spaces. Let $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ be the classes of tempered distributions and Schwartz functions, respectively. Let $\mathcal{F} = \{\|\cdot\|_{\alpha_i,\beta_i}\}$ be any finite collection of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ and

$$\mathcal{S}_{\mathcal{F}} = \{\psi \in \mathcal{S}(\mathbb{R}^n) : \|\psi\|_{\alpha_i,\beta_i} \le 1 \text{ for all } \|\cdot\|_{\alpha_i,\beta_i} \in \mathcal{F}\}.$$

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the grand maximal function is defined as

$$\mathcal{M}_{\mathcal{F}} f(x) = \sup_{\psi \in \mathcal{S}_{\mathcal{F}}} \sup_{t>0} |(f * \psi_t)(x)|,$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ for any $t > 0$.

Let $p \in (0, \infty)$. The Hardy space H^p consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{H^p} = \left(\int_{\mathbb{R}^n} |\mathcal{M}_{\mathcal{F}} f(x)|^p \, dx \right)^{1/p} < \infty.$$

The mapping properties of the fractional integral operators on Hardy spaces were obtained in [36]. A further extension of the results in [36] for the fractional type integral operators on Hardy spaces were established in [22].

Let $p \in (0, \infty)$ and $\omega : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function. The weighted Hardy space $H^p(\omega)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{H^p(\omega)} = \left(\int_{\mathbb{R}^n} |\mathcal{M}_{\mathcal{F}} f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty.$$

For the details of the weighted Hardy spaces, such as the atomic decomposition, the reader is referred to [13, 38].

We present the definition of the Muckenhoupt classes of weight functions in the following.

Definition 1. *For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_p weight if*

$$[\omega]_{A_p} = \sup_{B \in \mathbb{B}} \left(\frac{1}{|B|} \int_B \omega(x) \, dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p'}{p}} \, dx \right)^{\frac{p}{p'}} < \infty,$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_1 weight if there is a constant $C > 0$ such that for any $B \in \mathbb{B}$ we have

$$\frac{1}{|B|} \int_B \omega(y) \, dy \le C\omega(x), \quad \text{a.e. } x \in B.$$

The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \bigcup_{p \ge 1} A_p$.

Notice that we have $A_p \subseteq A_q$ whenever $1 \le p \le q$.

We now recall the weighted norm inequalities of the fractional integral operators on Hardy spaces. They are presented in terms of the mapping properties of the fractional integral operator on weighted Hardy spaces [24, 27, 36].

Theorem 1. *Let $0 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then $v^{q/p} \in A_\infty$ if and only if*

$$\|I_\alpha f\|_{H^q(v^{q/p})} \le C\|f\|_{H^p(v)}$$

for some $C > 0$.

The reader is referred to [36, Corollary 6.2 and Theorem 8.1] for the proof of the preceding theorem. Notice that the result presented in [36, Theorem 8.1] is for the case, when f is an (∞, N) atom. As stated at the beginning of [36, p.295], by using [36, Lemmas 2.1 and 2.2], the results in [36, Theorem 8.1] can be extended to obtain the boundedness of the fractional integral operators on the weighted Hardy spaces.

We now turn to the Lebesgue spaces with variable exponents.

Definition 2. *Let $p(\cdot):\mathbb{R}^n\to(0,\infty]$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ satisfying*

$$\|f\|_{L^{p(\cdot)}}=\inf\left\{\lambda>0:\rho_{p(\cdot)}(f/\lambda)\le 1\right\}<\infty,$$

where $\mathbb{R}^n_\infty=\{x\in\mathbb{R}^n:p(x)=\infty\}$ and

$$\rho_{p(\cdot)}(f)=\int_{\mathbb{R}^n\setminus\mathbb{R}^n_\infty}|f(x)|^{p(x)}\,dx+\operatorname*{ess\,sup}_{\mathbb{R}^n_\infty}|f(x)|.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$.

For $p(\cdot):\mathbb{R}^n\to[1,\infty]$, the Lebesgue space $L^{p(\cdot)}$ with variable exponent is a Banach function space [12, Theorem 3.2.13]. For simplicity, we refer the reader to [3, Chapter 1, Definitions 1.1 and 1.3] for the definition of Banach function space. In particular, the definition of Banach function space assures that $\chi_B\in L^{p(\cdot)}$ for all $B\in\mathbb{B}$.

For any Lebesgue measurable function $p(x):\mathbb{R}^n\to[1,\infty]$, define $p_-=\inf_{x\in\mathbb{R}^n}p(x)$ and $p_+=\sup_{x\in\mathbb{R}^n}p(x)$.

According to [12, Theorem 3.2.13], the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$.

Theorem 2. *Let $p(\cdot):\mathbb{R}^n\to(0,\infty]$ be a Lebesgue measurable function. If $1<p(x)<\infty$, $x\in\mathbb{R}^n$, then the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$, where p' satisfies $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.*

We call $p'(x)$ the conjugate function of $p(x)$.

In view of [12, Theorem 3.4.6], if $\sup_{x\in\mathbb{R}^n}p(x)<\infty$, then the dual space of $L^{p(\cdot)}$ equals to the associate space of $L^{p(\cdot)}$.

Definition 3. *Let $p(\cdot):\mathbb{R}^n\to[1,\infty]$ be a Lebesgue measurable function. We write $p(\cdot)\in\mathcal{B}$ if the Hardy-Littlewood maximal operator*

$$\mathrm{M}f(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|\,dy,$$

where the supremum is taken over all $B\in\mathbb{B}$ containing x, is bounded on $L^{p(\cdot)}$. We write $p(\cdot)\in\mathcal{B}'$ if $p'(\cdot)\in\mathcal{B}$.

Next, we recall a family of exponent functions for which the Hardy-Littlewood maximal function is bounded on the corresponding Lebesgue spaces with variable exponents.

Definition 4. *A continuous function g on $\mathbb{R}^n$ is locally log-Hölder continuous if there exists $c_{\log}>0$ such that*

$$|g(x)-g(y)|\le\frac{c_{\log}}{\log\left(e+1/|x-y|\right)},\quad\forall\,x,y\in\mathbb{R}^n.$$

We denote the class of locally log-Hölder continuous functions by $C^{\log}_{loc}(\mathbb{R}^n)$.

Furthermore, a continuous function is globally log-Hölder continuous if $g\in C^{\log}_{loc}(\mathbb{R}^n)$ and there exists $g_\infty\in\mathbb{R}$ such that

$$|g(x)-g_\infty|\le\frac{c_{\log}}{\log\left(e+|x|\right)},\quad\forall\,x\in\mathbb{R}^n.$$

The class of globally log-Hölder continuous functions is denoted by $C^{\log}(\mathbb{R}^n)$.

The Hardy-Littlewood maximal operator is bounded on the space $L^{p(\cdot)}$ provided that $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. For the proof of the subsequent theorem, the reader is referred to [12, Theorem 4.3.8].

Theorem 3. *If $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $1 < p_-$, then $p(\cdot) \in \mathcal{B}$.*

For more details on the Lebesgue spaces with variable exponents, the reader is referred to [12].

We now recall the definition of local Morrey spaces with variable exponents from [39, Definition 2.4].

Definition 5. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ and $u : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. The local Morrey space with variable exponent $LM_u^{p(\cdot)}$ consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{LM_u^{p(\cdot)}} = \sup_{r>0} \frac{1}{u(r)} \|\chi_{B(0,r)} f\|_{L^{p(\cdot)}} < \infty.$$

Whenever $p(\cdot) = p$, $1 \le p < \infty$, the local Morrey space with variable exponent reduces to the local Morrey space LM_u^p. The reader is referred to [5,7,8,15] for the details of local Morrey spaces.

The reader is referred to [18,39] for the boundedness of the geometrical maximal functions, the minimal function, the Carleson operator and the rough maximal functions on $LM_u^{p(\cdot)}$.

We restate the class of weight functions for local Morrey spaces with variable exponents given in [39, Definition 2.5].

Definition 6. *Let $q_0 \in (0,\infty)$ and $p(\cdot) : \mathbb{R}^n \to (0,\infty]$. We say that a Lebesgue measurable function $u : (0,\infty) \to (0,\infty)$ belongs to $\mathbb{LW}_{p(\cdot)}^{q_0}$ if there exists a constant $C > 0$ such that for any $r > 0$ the function u fulfills*

$$C \le u(r), \quad \forall r \ge 1, \tag{1}$$

$$\|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \le C u(r), \quad \forall r < 1, \tag{2}$$

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(0,r)}\|_{L^{p(\cdot)/q_0}}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^{p(\cdot)/q_0}}} \left(u\left(2^{j+1} r\right) \right)^{q_0} < C(u(r))^{q_0} \quad \text{for all } r > 0.$$

Whenever $q_0 = 1$, we write $\mathbb{LW}_{p(\cdot)} = \mathbb{LW}_{p(\cdot)}^1$. If $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ is a Lebesgue measurable function and u is increasing that satisfies (1) and (2), then $LM_u^{p(\cdot)}$ is ball Banach function space [39, Proposition 2.3 and Theorem 3.1]. For brevity, we refer the reader to [35] for the definition of ball Banach function space. Particularly, for any $B(z,r) \in \mathbb{B}$, we have $\chi_{B(z,r)} \in LM_u^{p(\cdot)}$.

We now study a pre-dual of $LM_u^{p(\cdot)}$. We recall the definition of local block spaces with variable exponents from [39, Definition 3.1].

Definition 7. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ and $u : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. A $b \in \mathcal{M}$ is a local $\left(u, L^{p(\cdot)}\right)$-block if it is supported in $B(0,r)$, $r > 0$, and*

$$\|b\|_{L^{p(\cdot)}} \le \frac{1}{u(r)}.$$

Define $\mathfrak{LB}_{u,p(\cdot)}$ *by*

$$\mathfrak{LB}_{u,p(\cdot)} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k \ : \ \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is a local } \left(u, L^{p(\cdot)}\right)\text{-block} \right\}.$$

The space $\mathfrak{LB}_{u,p(\cdot)}$ *is endowed with the norm*

$$\|f\|_{\mathfrak{LB}_{u,p(\cdot)}} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty} \lambda_k b_k \text{ a.e.} \right\}.$$

We call $\mathfrak{LB}_{u,p(\cdot)}$ *the local block space with variable exponent and denote the collection of the local* $\left(u, L^{p(\cdot)}\right)$*-block by* $\mathfrak{Lb}_{u,p(\cdot)}$.

The following results for $\mathfrak{LB}_{u,p(\cdot)}$ are given in [39]. It shows that the dual space of $\mathfrak{LB}_{u,p(\cdot)}$ is $LM_u^{p'(\cdot)}$.

Theorem 4. *Let* $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ *and* $u : (0,\infty) \to (0,\infty)$ *be Lebesgue measurable functions. We have*

$$\mathfrak{LB}^*_{u,p(\cdot)} = LM_u^{p'(\cdot)},$$

where $\mathfrak{LB}^*_{u,p(\cdot)}$ *denotes the dual space of* $\mathfrak{LB}_{u,p(\cdot)}$.

The reader is referred to [39, Theorem 3.1] for the proof of the preceding theorem.

In particular, we have the following propositions.

Proposition 1. *Let* $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ *and* $u : (0,\infty) \to (0,\infty)$ *be Lebesgue measurable functions. We have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le C\|f\|_{LM_u^{p(\cdot)}} \|g\|_{\mathfrak{LB}_{u,p'(\cdot)}}.$$

Proposition 2. *Let* f *be a Lebesgue measurable function. If*

$$\sup_{b\in\mathfrak{Lb}_{u,p'(\cdot)}} \left| \int_{\mathbb{R}^n} f(x)b(x)\, dx \right| < \infty,$$

then $f \in LM_u^{p(\cdot)}$.

Proposition 3. *Let* $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ *and* $u : (0,\infty) \to (0,\infty)$ *be Lebesgue measurable functions. There exists a constant* $C > 0$ *such that for any* $f \in LM_{p(\cdot),u}$ *we have*

$$C\|f\|_{LM_u^{p(\cdot)}} \le \sup_{b\in\mathfrak{Lb}_{u,p'(\cdot)}} \left| \int_{\mathbb{R}^n} f(x)b(x)\, dx \right| \le \|f\|_{LM_u^{p(\cdot)}}.$$

We have the boundedness of the Hardy-Littlewood maximal operator on $\mathfrak{LB}_{u,p(\cdot)}$ (see [39, Theorem 3.4]).

Theorem 5. *Let* $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ *and* $u : (0,\infty) \to (0,\infty)$ *be Lebesgue measurable functions. If* $p(\cdot) \in \mathcal{B}$ *and* $u \in \mathbb{LW}_{p'(\cdot)}$, *then the Hardy-Littlewood maximal operator* M *is bounded on* $\mathfrak{LB}_{u,p(\cdot)}$.

We recall the definition of the Hardy local Morrey spaces with variable exponents from [18, Definition 4.1].

Definition 8. *Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$ and $u:(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions. The Hardy local Morrey space with variable exponent $HLM_u^{p(\cdot)}$ consists of all $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\|f\|_{HLM_u^{p(\cdot)}}=\|\mathcal{M}_{\mathcal{F}}f\|_{LM_u^{p(\cdot)}}<\infty.$$

When $p(\cdot)=p$, $p\in(0,1]$, the Hardy local Morrey space with variable exponent becomes the Hardy local Morrey spaces $HLM_{p,u}$.

For the studies of the boundedness of the Calderón-Zygmund operators, the Littlewood-Paley functions and the maximal Bochner-Riesz means on Hardy local Morrey spaces with variable exponents, the reader may consult [18, Sections 4.1-4.3].

3 Main results

We use the extrapolation theory to obtain our main result introduced by J.L. Rubio de Francia [31–33]. An extrapolation theory for the local Morrey spaces with variable exponents is obtained in [39]. For the extrapolation theory given in [39], it requires the validity of the weighted norm inequalities for all weights belonging to A_1. This restriction has been relaxed in [18, Theorems 3.1 and 3.2]. The extrapolation theory given in [18, Theorems 3.1 and 3.2] just requires the validity of the weighted norm inequalities for a subset only. The following theorem further extends the extrapolation theory so that we can apply it to the fractional integral operators.

For any $\theta\in[1,\infty)$ and locally integrable function f, define

$$\mathrm{M}_\theta f=\left(\mathrm{M}\,|f|^\theta\right)^{1/\theta}.$$

Theorem 6. *Let $0<p_0\le q_0<\infty$, $p(\cdot),q(\cdot):\mathbb{R}^n\to(0,\infty)$, $u:(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions and $\mathcal{F}$ be a family of pairs of non-negative Lebesgue measurable functions. Suppose that p_0,q_0 and $p(\cdot),q(\cdot)$ satisfy $p_0<p_-\le p_+<\frac{p_0q_0}{q_0-p_0}$, $q_0<q_-$ and*

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{1}{p_0}-\frac{1}{q_0}.$$

Suppose there exist $\theta\in[1,\infty)$ and $C_0>0$ such that $(q(\cdot)/q_0)'/\theta\in\mathcal{B}$ and for any $r>0$

$$\sum_{j=1}^{\infty}\left(\frac{|B(0,r)|}{|B(0,2^{j+1}r)|}\right)^{p_0/q_0\theta}\frac{\|\chi_{B(0,2^{j+1}r)}\|_{L(p(\cdot)/p_0)'}}{\|\chi_{B(0,r)}\|_{L(p(\cdot)/p_0)'}}\frac{u\left(2^{j+1}r\right)^{p_0}}{u(r)^{p_0}}<C_0. \tag{3}$$

If for every

$$\omega\in\left\{\mathrm{M}_\theta h: h\in\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}\right\}$$

and $(f,g)\in\mathcal{F}$, we have

$$\left(\int_{\mathbb{R}^n}f(x)^{q_0}\omega(x)\,dx\right)^{1/q_0}\le C\left(\int_{\mathbb{R}^n}g(x)^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0}<\infty, \tag{4}$$

where C is independent of f and g, then there exists a constant $C>0$ such that for any $(f,g)\in\mathcal{F}$ with $g\in LM_u^{p(\cdot)}$, we have $f\in LM_u^{p(\cdot)}$ and

$$\|f\|_{LM_u^{q(\cdot)}}\le C\|g\|_{LM_u^{p(\cdot)}}.$$

Proof. Since $\left(q(\cdot)/q_0\right)'/\theta \in \mathcal{B}$, [16, Lemma 3.2] ensures that

$$\frac{\|\chi_{B(0,r)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}} \le C\frac{|B(0,r)|}{\left|B\left(0,2^{j+1}r\right)\right|}\frac{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{(q(\cdot)/q_0)'/\theta}}}{\|\chi_{B(0,r)}\|_{L^{(q(\cdot)/q_0)'/\theta}}}$$
$$= C\frac{|B(0,r)|}{\left|B\left(0,2^{j+1}r\right)\right|}\left(\frac{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{\left(q(\cdot)/q_0\right)'}}}{\|\chi_{B(0,r)}\|_{L^{\left(q(\cdot)/q_0\right)'}}}\right)^{\theta}$$

for some $C>0$. As

$$\left(p(\cdot)/p_0\right)' = \frac{p(\cdot)}{p(\cdot)-p_0} = \frac{q_0}{p_0}\frac{q(\cdot)}{q(\cdot)-q_0} = \frac{q_0}{p_0}\left(q(\cdot)/q_0\right)', \tag{5}$$

we get

$$\left(\frac{\|\chi_{B(0,r)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}\right)^{p_0/q_0\theta} \le C\left(\frac{|B(0,r)|}{\left|B\left(0,2^{j+1}r\right)\right|}\right)^{p_0/q_0\theta}\left(\frac{\|\chi_{B(0,2^{j+1}r)}\|_{L^{(q(\cdot)/q_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(q(\cdot)/q_0)'}}}\right)^{p_0/q_0}$$
$$= C\left(\frac{|B(0,r)|}{\left|B\left(0,2^{j+1}r\right)\right|}\right)^{p_0/q_0\theta}\frac{\|\chi_{B(0,2^{j+1}r)}\|_{L^{(q_0/p_0)(q(\cdot)/q_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(q_0/p_0)(q(\cdot)/q_0)'}}}$$
$$= C\left(\frac{|B(0,r)|}{\left|B\left(0,2^{j+1}r\right)\right|}\right)^{p_0/q_0\theta}\frac{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{(p(\cdot)/p_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(p(\cdot)/p_0)'}}}.$$

Since $\theta\in[1,\infty)$ and $p_0\le q_0$, we find that

$$\left(\sum_{j=1}^{\infty}\frac{\|\chi_{B(0,r)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}\frac{u\left(2^{j+1}r\right)^{q_0\theta}}{u(r)^{q_0\theta}}\right)^{p_0/q_0\theta}$$
$$\le \sum_{j=1}^{\infty}\left(\frac{\|\chi_{B(0,r)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}\frac{u(^{j+1}r)^{q_0\theta}}{u(r)^{q_0\theta}}\right)^{p_0/q_0\theta} \tag{6}$$
$$= \sum_{j=1}^{\infty}\left(\frac{|B(0,r)|}{\left|B\left(0,2^{j+1}r\right)\right|}\right)^{p_0/q_0\theta}\frac{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{(p(\cdot)/p_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(p(\cdot)/p_0)'}}}\frac{u\left(2^{j+1}r\right)^{p_0}}{u(r)^{p_0}}.$$

For any $h\in\mathfrak{Lb}_{u^{q_0},L^{(q(\cdot)/q_0)'}}$ with $\operatorname{supp} h\subset B(0,r)$, we have $|h|^{\theta}\in\mathfrak{Lb}_{u^{q_0\theta},L^{(q(\cdot)/q_0)'/\theta}}$. According to (3) and (6), we get $u^{q_0\theta}\in\mathbb{LW}_{((q(\cdot)/q_0)'/\theta)'}$, [10, Theorem 3.1] guarantees that there exist a family of blocks $\{d_j\}_{j=1}^{\infty}\subset\mathfrak{Lb}_{u^{q_0\theta},L^{(q(\cdot)/q_0)'/\theta}}$ and

$$\gamma_j = \frac{\|\chi_{B(0,r)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}{\|\chi_{B\left(0,2^{j+1}r\right)}\|_{L^{((q(\cdot)/q_0)'/\theta)'}}}\frac{u\left(2^{j+1}r\right)^{q_0\theta}}{u(r)^{q_0\theta}},$$

such that

$$\mathrm{M}\left(|h|^{\theta}\right) = \sum_{j=1}^{\infty}\gamma_j d_j.$$

Since $\theta\in[1,\infty)$ and $p_0\le q_0$, we find that

$$(\mathrm{M}_{\theta}\,h)^{p_0/q_0} \le \sum_{j=1}^{\infty}|\gamma_j|^{p_0/q_0\theta}|d_j|^{p_0/q_0\theta}.$$

In view of (3) and (6), $\left\|\left\{|\gamma_j|^{p_0/q_0\theta}\right\}_{j=1}^{\infty}\right\|_{\ell^1} \leq C_0$. Since $\{d_j\}_{j=1}^{\infty} \subset \mathfrak{Lb}_{u^{q_0\theta},L^{(q(\cdot)/q_0)'/\theta}}$, (5) asserts $\left\{|d_j|^{p_0/q_0\theta}\right\}_{j=1}^{\infty} \subset \mathfrak{Lb}_{u^{p_0},L^{(p(\cdot)/p_0)'}}$. Definition 7 yields

$$\left\|(\mathrm{M}_\theta h)^{p_0/q_0}\right\|_{\mathfrak{LB}_{u^{p_0},L^{(p(\cdot)/p_0)'}}} \leq C_0. \tag{7}$$

Since $g \in \mathcal{M}_{p(\cdot),u}$, Proposition 1, (4) and (7) guarantee that

$$\int_{\mathbb{R}^n} |f(x)|^{q_0}|h(x)|\,dx \leq C\int_{\mathbb{R}^n} |f(x)|^{q_0}\,\mathrm{M}_\theta h(x)\,dx \leq \left(\int_{\mathbb{R}^n} |g(x)|^{p_0}\left(\mathrm{M}_\theta h(x)\right)^{p_0/q_0}dx\right)^{q_0/p_0}$$
$$\leq \left\||g|^{p_0}\right\|_{LM_{p(\cdot)/p_0,u^{p_0}}}^{q_0/p_0}\left\|(\mathrm{M}_\theta h)^{p_0/q_0}\right\|_{\mathfrak{LB}_{u^{p_0},L^{(p(\cdot)/p_0)'}}}^{q_0/p_0} \leq C\|g\|_{LM_{p(\cdot),u}}^{q_0}.$$

By taking the supremum over $h \in \mathfrak{Lb}_{u^{q_0},L^{(q(\cdot)/q_0)'}}$, we get

$$\sup_{h\in\mathfrak{Lb}_{u^{q_0},L^{(q(\cdot)/q_0)'}}}\int_{\mathbb{R}^n} |f(x)|^{q_0}|h(x)|\,dx \leq C\|g\|_{LM_{p(\cdot),u}}^{q_0},$$

Proposition 2 yields $|f|^{q_0} \in LM_{q(\cdot)/q_0,u^{q_0}}$. Consequently, $f \in LM_{q(\cdot),u}$. Finally, Proposition 3 assures that

$$\|f\|_{LM_{q(\cdot),u}} = \left\||f|^{q_0}\right\|_{LM_{q(\cdot)/q_0,u^{q_0}}}^{1/q_0} \leq C\|g\|_{LM_{p(\cdot),u}}.$$

□

We use the above result to establish the mapping properties of the fractional integral operators on the Hardy local Morrey spaces with variable exponents. Theorem 6 can also be modified to apply to the fractional geometric maximal functions [21].

We now ready to establish the mapping properties of the fractional integral operators on the Hardy local Morrey spaces with variable exponents.

Theorem 7. *Let $\alpha > 0$, $u : (0,\infty)$ be a Lebesgue measurable function, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \leq p_+ < \infty$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \mathbb{R}^n. \tag{8}$$

If there exist $\lambda \in \left[0, \frac{1}{p_+} - \frac{\alpha}{n}\right)$ and $C > 0$ such that for any $r > 0$,

$$u(2^j r) \leq C2^{jn\lambda}u(r), \tag{9}$$

then there is a constant $C > 0$ such that

$$\|I_\alpha f\|_{HLM_u^{q(\cdot)}} \leq C\|f\|_{HLM_u^{p(\cdot)}} \quad \forall f \in HLM_u^{p(\cdot)}.$$

Proof. According to (8), we have $\frac{1}{p_+} > \frac{\alpha}{n}$. Thus, $\lambda < \frac{1}{p_+} - \frac{\alpha}{n}$ guarantees that there exists a $\theta > 1$, such that for any fixed $q_0 \in (0, q_-)$, we have

$$q_0\frac{\alpha}{n} + 1 - \frac{q_0}{p_+} + q_0\lambda < 1. \tag{10}$$

Let p_0 be selected such that

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}.$$

Therefore,

$$\frac{1}{p_-} - \frac{1}{q_-} = \frac{\alpha}{n} = \frac{1}{p_0} - \frac{1}{q_0}.$$

Since $q_0 \in (0, q_-)$, we find that $p_0 \in (0, p_-)$. Additionally, $\frac{1}{p_+} > \frac{1}{p_+} - \frac{1}{q_+} = \frac{1}{p_0} - \frac{1}{q_0}$, which gives $p_+ < \frac{p_0 q_0}{q_0 - p_0}$.

Furthermore, we find that $q_0 \frac{\alpha}{n} + 1 = \frac{q_0}{p_0}$. Consequently, (10) gives

$$\frac{q_0}{p_0} - \frac{q_0}{p_+} + q_0\lambda < 1.$$

That is,

$$1 - \frac{p_0}{p_+} + \lambda p_0 < \frac{p_0}{q_0}.$$

In view of the above inequality and [11, Theorem 8.1], we are allowed to select a $\theta > 1$, such that $\big(q(\cdot)/q_0\big)'/\theta \in \mathcal{B}$ and

$$1 - \frac{p_0}{p_+} + \lambda p_0 < \frac{p_0}{q_0\theta}. \tag{11}$$

Since $p(\cdot)/p_- \in C^{\log}(\mathbb{R}^n)$, [12, Corollary 4.5.9] ensures that whenever $\big|B(0,r)\big| > 1$ or $\big|B\big(0, 2^{j+1}r\big)\big| < 1$, we have

$$\frac{\|\chi_{B(0,2^{j+1}r)}\|_{L^{(p(\cdot)/p_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(p(\cdot)/p_0)'}}} \leq C2^{jn/(p_+/p_0)'}. \tag{12}$$

In addition, [12, Corollary 4.5.9] also asserts that whenever $\big|B(0,r)\big| < 1$ and $\big|B\big(0,2^{j+1}r\big)\big| > 1$, we have

$$\frac{\|\chi_{B(0,2^{j+1}r)}\|_{L^{(p(\cdot)/p_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(p(\cdot)/p_0)'}}} \leq C\frac{\big|B\left(0,2^{j+1}r\right)\big|^{1/(p_\infty/p_0)'}}{\big|B(0,r)\big|^{1/(p(0)/p_0)'}},$$

where $(p(0)/p_0)'$ denotes $(p(x)/p_0)'|_{x=0}$. We have

$$\begin{aligned}
\big|B\big(0,2^{j+1}r\big)\big|^{1/(p_\infty/p_0)'} &\leq \big|B\big(0,2^{j+1}r\big)\big|^{1/((p(\cdot)/p_0)')_-}, \\
\big|B(0,r)\big|^{1/(p(0)/p_0)'} &\geq \big|B(0,r)\big|^{1/((p(\cdot)/p_0)')_-}.
\end{aligned}$$

That is, in this case, we also have (12).

Consequently, (11) asserts that

$$\begin{aligned}
\sum_{j=1}^{\infty}\left(\frac{\big|B(0,r)\big|}{\big|B\big(0,2^{j+1}r\big)\big|}\right)^{p_0/q_0\theta} & \frac{\|\chi_{B(0,2^{j+1}r)}\|_{L^{(p(\cdot)/p_0)'}}}{\|\chi_{B(0,r)}\|_{L^{(p(\cdot)/p_0)'}}}\frac{u\big(2^{j+1}r\big)^{p_0}}{u(r)^{p_0}} \\
&\leq C\sum_{j=1}^{\infty} 2^{-jnp_0/q_0\theta}2^{jn(p_+/p_0)'}2^{jn\lambda p_0} \\
&\leq C\sum_{j=1}^{\infty} 2^{jn\big((-p_0/q_0\theta)+1-(p_0/p_+)+\lambda p_0\big)} < C_0
\end{aligned}$$

for some $C_0 > 0$ independent of $r > 0$. That is, (3) is fulfilled.

For any $h \in \mathfrak{Lb}_{u^{q_0},L^{(q(\cdot)/q_0)'}}$ and $f \in HLM_u^{p(\cdot)}$, (7) yields

$$\left(\int_{\mathbb{R}^n} |\mathcal{M}_{\mathcal{F}} f(x)|^{p_0} \left(\mathrm{M}_\theta h(x)\right)^{p_0/q_0} dx\right)^{q_0/p_0}$$
$$\leq \left\| |\mathcal{M}_{\mathcal{F}} f|^{p_0} \right\|_{LM_{p(\cdot)/p_0,u^{p_0}}}^{q_0/p_0} \left\| (\mathrm{M}_\theta h)^{p_0/q_0} \right\|_{\mathfrak{LB}_{u^{p_0},L^{(p(\cdot)/p_0)'}}}^{q_0/p_0}$$
$$\leq C \|LM_{\mathcal{F}} f\|_{LM_{p(\cdot),u}}^{q_0} = C\|f\|_{HLM_u^{p(\cdot)}}^{q_0}.$$

Hence,

$$HLM_u^{p(\cdot)} \hookrightarrow \bigcap_{h \in \mathfrak{Lb}_{u^{q_0},L^{(q(\cdot)/q_0)'}}} H^{p_0}\left((\mathrm{M}_\theta h)^{p_0/q_0}\right). \tag{13}$$

Since $\theta > 1$, [14, Theorem 9.2.8] guarantees that $\mathrm{M}_\theta h(x) \in A_1 \subseteq A_\infty$. By applying Theorem 1 with $v = \left(\mathrm{M}_\theta h(x)\right)^{p_0/q_0}$, we get

$$\left(\int_{\mathbb{R}^n} \left(\mathcal{M}_{\mathcal{F}} I_\alpha f(x)\right)^{q_0} \mathrm{M}_\theta h(x)\, dx\right)^{1/q_0} \leq C_0 \left(\int_{\mathbb{R}^n} |\mathcal{M}_{\mathcal{F}} f(x)|^{p_0} \left(\mathrm{M}_\theta h(x)\right)^{p_0/q_0} dx\right)^{1/p_0}. \tag{14}$$

Since (13) and (14) guarantee that (4) is valid for the set

$$\mathcal{F}_0 = \left\{ (\mathcal{M}_{\mathcal{F}} I_\alpha f, \mathcal{M}_F f) : f \in HLM_u^{p(\cdot)} \right\},$$

Theorem 6 yields a constant $C > 0$ such that for any $f \in HLM_u^{p(\cdot)}$ we have

$$\|I_\alpha f\|_{HLM_u^{q(\cdot)}} = \|\mathcal{M}_{\mathcal{F}} I_\alpha f\|_{LM_u^{q(\cdot)}} \leq C\|\mathcal{M}_{\mathcal{F}} f\|_{LM_u^{p(\cdot)}} = C\|f\|_{HLM_u^{p(\cdot)}}.$$

□

Particularly, when $\alpha > 0$, $p(\cdot) = p$, $p \in (0,1]$, is a constant function and u satisfies (9) for some $\lambda \in \left[0, \frac{1}{p} - \frac{\alpha}{n}\right)$, then $I_\alpha : HLM_{p,u} \to HLM_{q,u}$ is bounded, where $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

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Received 03.09.2021
Revised 25.04.2022

Хо К.-П. *Дробові інтегральні оператори на локальних просторах Харді-Моррі зі змінними показниками* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 190–202.

Ми встановлюємо властивості відображення дробових інтегральних операторів на локальних просторах Харді-Моррі зі змінними показниками за допомогою теорії екстраполяції. Локальні простори Моррі зі змінними показниками є узагальненнями та розширеннями локальних просторів Моррі, просторів Лебега зі змінними показниками і просторів Моррі зі змінними показниками. Локальні простори Харді-Моррі зі змінними показниками – це простори Харді, побудовані на локальних просторах Моррі зі змінними показниками. Наш головний результат розширює та узагальнює властивості відображення дробових інтегральних операторів на просторах Харді, локальних просторах Моррі, просторах Харді зі змінними показниками і локальних просторах Моррі зі змінними показниками. Ми отримуємо наш головний результат, розширюючи теорію екстраполяції Х.Л. Рубіо де Франчіа до локальних просторів Моррі зі змінними показниками. Цей метод спочатку був розроблений Х.Л. Рубіо де Франчіа для зважених просторів Лебега, а нещодавно його було розширено до ряду функціональних просторів, таких як простори Моррі зі змінними показниками, локальні простори Моррі зі змінними показниками і простори Моррі-Банаха. Далі ми поширюємо його на локальні простори Моррі зі змінними показниками. Використовуючи властивості відображення дробових інтегральних операторів на зважених просторах Харді, ми встановлюємо властивості відображення дробових інтегральних операторів на локальних просторах Харді-Моррі зі змінними показниками.

Ключові слова і фрази: дробовий інтегральний оператор, простір Харді, локальний простір Моррі, змінний показник.